# Above, below and beyond Brownian motion

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Brownian motion represents simple diffusion random walk processes. More complex random walk processes also can occur when probability distributions describing the random jump distances and times have infinite moments. We explore the manner in which these distributions can arise and how they underlie various scaling laws that play an important role in both random and deterministic systems. © *1999 American Association of Physics Teachers.*

## I. INTRODUCTION

If you sprinkle powdered charcoal on the surface of alcohol and look at it under a microscope, you will see the charcoal particles undergoing a random walk. This motion is due to the alcohol molecules colliding with the larger charcoal grains. This experiment was first reported by the Dutch physician Jan Ingenhausz in 1785, who is best known as the discoverer of photosynthesis. The observed random walk is known as Brownian motion, after its extensive investigation by Robert Brown published in 1828. Brown also is known for making the first observation of a plant cell nucleus. Brownian motion was mysterious in those early days before the existence of atoms was demonstrated, and it was not clear why the Brownian particles should jump seemingly on their own. The eventual explanation came from Albert Einstein in 1905,[1] but he did not refer to it as Brownian motion, because he had not yet seen Brown's papers. Einstein was able to determine the mean square displacement of a Brownian particle in terms of Avogadro's number. Jean Perrin won the 1926 Nobel Prize in physics for determining Avogadro's number in this manner.[2]

Since 1905, Brownian motion has became the canonical example of a random process.[3] Actually, Louis Batchelier in his 1900 Ph.D. thesis on stock market fluctuations independently derived several mathematical properties of Brownian motion, including the equation for the probability $P(x,t)$ for the position $x$ of a Brownian random walker at time $t$, when the walker starts at the origin at time $t=0$. The equation for $P(x,t)$ in one dimension is given by

$$\frac{\partial P(x,t)}{\partial t} = D \frac{\partial^2 P(x,t)}{\partial^2 x},\tag{1}$$

with the Gaussian solution

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt},\tag{2}$$

and the mean square displacement

$$\langle x^2(t)\rangle = 2Dt.\tag{3}$$

Equation (1), the diffusion equation, was already well known as Fourier's heat law, and Bachelier was amazed that probability could diffuse in the same manner as heat. Note that the diffusion constant $D$ has units of $[x^2]/[t]$.

The diffusion equation does not provide information on the shape of a typical particle trajectory. If we look at small displacements $\Delta x$ in a small time $\Delta t$, then $D$ is given by the limit $(\Delta x)^2/\Delta t$ as both $\Delta x$ and $\Delta t$ go to zero. Because $D$ is finite, $\Delta x/\Delta t$ must be infinite, which leads us to conclude that the velocity, $\Delta x/\Delta t$, of a Brownian particle (the derivative along a Brownian trajectory curve) is everywhere infinite. Therefore, a Brownian trajectory is infinitely jagged and care is needed to mathematically analyze Brownian trajectories. Wiener proved that the distance between any two points on a Brownian trajectory is infinite, because the trajectory is actually two-dimensional, and not a simple curved line. We can avoid these mathematical difficulties of the Brownian trajectory if we consider the random walk motion to take place on a periodic lattice with jumps occurring at a regular rate. In this case the trajectory is a connected path of straight line pieces. In the long time (many jump) limit, the solution for the random walk on a lattice approaches the behavior of Eqs. (1)–(3).

The purpose of this article is to discuss random walks for which Eqs. (1)–(3) are no longer applicable. This situation can occur if the walker waits for very long times between jumps (fractal time), or if the jumps are of very large distances (Levy flights). For fractal time random walks, the mean square displacements are slower than Brownian motion; this type of random walk refers to the ''below'' Brownian motion in the title. For Levy flights we will introduce a velocity for the jumps that is related to the jump distance. This velocity will allow us to discuss turbulent diffusion for which the mean square displacement grows as the third power of time, in contrast to the first power for Brownian motion.[4] This behavior refers to the ''above'' Brownian motion in the title. We will also discuss trajectories in deterministic systems where nonlinearities create a wide distribution of trajectory lengths. The description of these deterministic systems fits within the framework of Levy flight type random walks. Because these systems are deterministic and not random, we refer to them in the title as ''beyond'' Brownian motion.[5]

## II. LONG-TAILED DISTRIBUTIONS IN PHYSICS

In the early 1700's a coin tossing problem was posed by Nicolas Bernoulli that yielded a surprising result. The problem was later discussed by Daniel Bernoulli in the Commen-

tary of the St. Petersburg Academy, and became known as the St. Petersburg paradox.[6] The paradox involves calculating the average winnings in a game of chance. The game is simple to describe. Flip a coin. If a head comes up, you win one coin and the game ends. This case occurs with probability 1/2. If you get a tail, then flip again. Continue flipping until a head comes up for the first time. If you obtain precisely $N$ tails prior to the first head, you are rewarded with a win of $2^N$ coins. The probability for this event is $(1/2)^{N+1}$. So you can win one coin with probability 1/2, two coins with probability 1/4, etc. We can see a regular pattern develop where winning an order of magnitude more (in base 2) occurs with an order of magnitude less probability. The mean winning is $[1 \times (1/2) + 2 \times (1/4) + 4 \times (1/8) + \cdots]$. This sum adds 1/2 an infinite number of times, making the mean winning infinite. The result was considered a paradox because the notion of a probability distribution with an infinite first moment seemed ill posed. (The probabilities for all possible events adds up to unity, so that is not the problem.) The paradox arises because we are trying to determine the mean of a distribution that does not possess one. In particular, the lack of a finite first moment means that we cannot determine an ante to make the game fair. A banker would need to have an infinite ante because this amount is her expected loss. A player would prefer a small ante because 1/2 of the time he will win one coin, 3/4 of the time he will win two coins or fewer, 7/8 of the time four or fewer coins, 15/16 of the time eight or fewer coins, etc. Much of the early commentary on this paradox centered around limiting the number of times a coin could be tossed, and thus limiting the amount of winnings.

It is not difficult to produce random variables whose probability densities have long tails with infinite moments. We give a few examples in the following. If the distribution of a random variable peaks at the origin, then the distribution of the inverse of this variable may have a slow decay at large values of the variable. For example, let the random variable $x$ have a Gaussian probability density $g(x)$ with zero mean and unit variance:

$$g(x) = (2\pi)^{-1} e^{-x^2/2}. \tag{4}$$

The random variable $y = 1/x$ has the probability density $f(y)$ which is related to $g(x)$ by

$$f(y) dy = g(x) dx = (2\pi)^{-1} e^{-1/2y^2} \frac{dy}{y^2}. \tag{5}$$

The distribution $f(y)$ in Eq. (5) has an infinite variance because $f(y)$ goes to zero for large $y$ as $1/y^2$.

The Gaussian distribution arises as the probability limit distribution for a sum of identically distributed random variables. The lognormal distribution arises as a probability distribution for a product of identically distributed random variables. That is, the lognormal is a Gaussian distribution for the variable $\log(x)$. If $m(x)$ is the lognormal probability distribution (with $x$ measured in units of $\langle x \rangle$) with unit variance and $g(x)$ is the Gaussian, then

$$m(x) dx = g(\log x) d(\log x) = g(\log x) \frac{dx}{x}, \tag{6}$$

and hence

$$m(x) = \frac{1}{x} \frac{1}{\sqrt{2\pi}} e^{-(\log x)^2/2}. \tag{7}$$

We see from Eq. (7) that there will be some intermediate range of $x$ values for which $m(x)$ behaves like it is slowly decaying as $1/x$.

Next consider a light source at point $A$ that releases photons that travel in straight lines through random angles $\theta$ and strike a wall one unit away in the $x$-direction and a distance $L$ units away in the $y$-direction. If the angle $\theta$ has a distribution $f(\theta)$ uniformly distributed between $-\pi/2$ and $\pi/2$, then $f(\theta) = 1/\pi$. The distribution of angles $\theta$ will induce a distribution of hitting points $g(y)$ along the wall given by

$$f(\theta) d\theta = \frac{1}{\pi} d\theta = g(y) dy, \tag{8}$$

where

$$\theta = \tan^{-1} y \quad \text{and} \quad \frac{d\theta}{dy} = \frac{1}{1+y^2}. \tag{9}$$

We can use Eqs. (8)–(9) to solve for $g(y)$ as

$$g(y) = \frac{1}{\pi} \frac{1}{1+y^2}. \tag{10}$$

The probability distribution $g(y)$ is the Cauchy distribution, which has an infinite second moment.

As an example of a long-tailed distribution in time, consider a particle in a potential well of height $V$. Suppose that the time the particle spends in the well is given by the Arrhenius law,

$$\tau = \nu^{-1} e^{V/kT}, \tag{11}$$

where $\nu$ is a frequency, $k$ is Boltzmann's constant, and $T$ is the temperature. Let the barrier height $V$ be a random variable governed by the probability density $f(V)$, where

$$f(V) = e^{-V/V_0}, \tag{12}$$

that is, $V_0$ is the width of the distribution. This distribution of barrier heights induces a distribution of trapping times,

$$\psi(\tau) = f(V) \frac{dV}{d\tau}. \tag{13}$$

Using Eqs. (11)–(13) we find the distribution of trapping times to be[7]

$$\psi(\tau) = kT\nu^{-\beta} \tau^{-(1+\beta)}, \tag{14}$$

where $\beta = kT/V_0$.

We are interested in the long time behavior of $\psi(\tau)$. If there is a minimum barrier height greater than zero, then $\tau$ will always be greater than zero, and we can ignore the short time singularity in Eq. (14). A value of $\beta > 0$ ensures that the probability is normalized. If $\beta < 1$, the mean time for hopping over a barrier is infinite. This case does not mean that every hop has an infinite waiting time. It just means that waiting times of all possible values arise, such that none of them dominate and there is no characteristic waiting time. This case is like the St. Petersburg paradox where not every player wins an infinite amount of coins, although the mean winnings are infinite. If $\beta \geq 1$, the mean jump time is finite and the jump events occur with a finite average frequency.

The above examples demonstrate that it is not difficult to obtain probability distributions with infinite moments.

## III. FRACTAL TIME

Let us look at the case of infinite $\langle t \rangle$ in a more general context than Eq. (14). Consider the Laplace transform of the waiting-time density,

$$\psi^*(s) \equiv \int_0^\infty \psi(t) e^{-st} dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!} s^n \langle t^n \rangle, \quad (15)$$

where $\langle t^n \rangle$ is the $n$th moment of $\psi(t)$. This expansion holds only if all the moments of $\psi(t)$ are finite. In this case,

$$\psi^*(s) \approx 1 - s\langle t \rangle \approx \frac{1}{1 + s\langle t \rangle} \quad \text{(for small s),} \quad (16)$$

and at long times,

$$\psi(t) \approx \frac{1}{\langle t \rangle} e^{-t/\langle t \rangle}. \quad (17)$$

The probability density for making $n$ jumps in time $t$ would be

$$\psi_n(t) = \frac{1}{n!\langle t \rangle^n} e^{-t/\langle t \rangle}, \quad (18)$$

which is the Poisson distribution.

Let us consider a case where the first and all higher moments of the the distribution are infinite, and the Poisson distribution does not apply. We choose the form of $\psi(t)$ as

$$\psi(t) = \frac{1-q}{q} \sum_{j=1}^\infty q^j \lambda^j e^{-\lambda^j t} \quad (\lambda < q < 1). \quad (19)$$

Equation (19) for $\psi(t)$ satisfies a scaling form, with an order of magnitude longer waiting time (in base $1/\lambda$) occurring an order of magnitude less often (in base $1/q$). This form of $\psi(t)$ allows for a hierarchy of weighted possibilities including extreme events (very long waiting times). This form can be a good description of charge hopping waiting times in glassy materials.[7,8] The waiting-time distribution in Eq. (19) is easily checked to be normalized to unity, but the condition $\lambda < q < 1$ leads to $\langle t \rangle$ being infinite, that is,

$$\langle t \rangle = \frac{1-q}{q} \sum_{j=1}^\infty \left(\frac{q}{\lambda}\right)^j = \infty \quad (\lambda < q < 1). \quad (20)$$

If this condition is not satisfied, then the long waiting times are not weighted heavily enough to make the first moment of the distribution infinite. In that case the waiting-time distribution would be characterized by a finite mean waiting time, and the Poisson distribution would become valid at times much larger than the mean waiting time.

How can we proceed with a power series expansion of $\psi^*(s)$? We calculate the Laplace transform of Eq. (19) term by term to obtain $\psi^*(s)$ as

$$\psi^*(s) = \frac{1-q}{q} \sum_{j=1}^\infty \frac{(q\lambda)^j}{s + \lambda^j}. \quad (21)$$

This scaling relation satisfies

$$\psi^*(s) = q\psi^*\left(\frac{s}{\lambda}\right) + \frac{\lambda(1-q)}{s+\lambda}, \quad (22)$$

and that allows us to determine the noninteger exponent in the expansion of $\psi^*(s)$. A noninteger exponent must come from the homogeneous part of the equation, $\psi^*(s) = q\psi^*(s/\lambda)$, which has a solution of the form

$$\psi^*(s) \approx s^\beta, \quad (23)$$

with

$$\beta = \frac{\ln q}{\ln \lambda}. \quad (24)$$

The asymptotic behavior of $\psi^*(s)$ for small $s$ and large $t$ is given by

$$\psi^*(s) \approx 1 - s^\beta + O(s) \quad (s \to 0), \quad (25)$$

and as in Eq. (14),

$$\psi(t) \approx t^{-1-\beta} \quad (t \to \infty). \quad (26)$$

The first term in Eq. (25) appears because at $s=0$, $\psi^*(s) = 1$, which reflects the normalization of the probability to unity. If $\beta > 1$, then $\langle t \rangle$ is finite, and the $s$ term dominates the $s^\beta$ term in Eq. (25) for small $s$, so that the standard Poisson behavior returns when $\langle t \rangle$ is finite. The condition, $\lambda < q < 1$, which makes $\langle t \rangle$ infinite, also keeps $\beta < 1$. The expression for $\psi(t)$ in Eq. (25) with $\beta < 1$ represents a *fractal time* random process, because if we draw points on a time axis when jumps occur, the set of points would look like a random Cantor set with fractal dimension $\beta$.[6,8] The number of jumps grows in a time $t$ as $t^\beta$ and not as $t$. This dependence implies that the jumps do not occur at a well-defined rate. Such behavior can occur if the jump times fall in a hierarchy of clusters that does not possess a well-defined mean waiting time.

In analyzing random walks we must account for all possible jumps and the timing of these jumps.[9] The terms in Laplace space,

$$\frac{1}{1 - \psi^*(s)} = 1 + \psi^*(s) + [\psi^*(s)]^2 + \cdots + [\psi^*(s)]^n$$

$$+ \cdots, \quad (27)$$

account for $1, 2, \ldots, n, \ldots$, jumps occurring in a time $t$. From Eq. (25), $1/[1 - \psi^*(s)]$ will produce a factor of $s^{-\beta}$ if $\langle t \rangle$ is infinite. The fractal time probability is the key to understanding how the exponent $\beta$ arises in the dispersive transport that is discussed in the next section.

## IV. DISPERSIVE TRANSPORT

The mechanism of fractal time has been used to describe charges (electrons or holes) moving in glassy materials, specifically in thin amorphous semiconductor films used in xerography.[7,8] The charges are trapped in a random environment and their release is governed by a long tailed distribution as in Eq. (26). The current generated by the motion of the charges is measured when the sample is placed in an electric field. In an electric field the charges tend to move in one direction, that is, they undergo a biased random walk in the direction of the net force on them. The randomness in the charge motion comes from the variability of when the walker jumps, rather than the jump length itself. Each jump has a mean distance $\langle d(E) \rangle$, which we assume is proportional to

the electric field $E$. The mean distance covered after a time $t$ can be calculated for a fractal time random walk governed by an exponent $\beta$ and is given by[7,8]

$$\langle l(t)\rangle \approx \langle d(E)\rangle t^\beta \approx Et^\beta. \tag{28}$$

The exponent $\beta$ would be unity for a standard random walk where jumps occur at a regular rate. For $\beta<1$, this type of motion is called ''dispersive transport.''[7] It is slower than Brownian motion because the random walkers have an infinite mean waiting time.

If the thin film has a length $L$, then at some time $T$ the mean position of a packet of charges will be equal to $L$. When the mean position of a charge is $L$, we have from Eq. (28),

$$L \approx ET^\beta, \tag{29}$$

or

$$\frac{1}{T} \approx \left(\frac{E}{L}\right)^{1/\beta}. \tag{30}$$

The nonlinear dependence of $T$ on $L$ and $E$ in Eq. (30) was first viewed as paradoxical. Eventually it was appreciated that this behavior was a consequence of deep traps that induce the long waiting times described by the fractal time waiting distribution.

## V. SLOW RELAXATION

Slow relaxation in glassy materials has been discussed in the context of a model of fractal time transport. Consider the problem of dielectric relaxation of a glass involving frozen-in dipoles that can be relaxed only when it is hit by a mobile defect. This problem involves the first passage time of random walkers (defects) for reaching the origin (the frozen-in dipole). Suppose that there are $V$ lattice sites and $N$ walkers initially randomly distributed among these sites, not including the origin. The probability density $\Phi(t)$ that none of the walkers has reached the origin by time $t$ is given by

$$\Phi(t) = \left[1 - \frac{1}{V}\int d\mathbf{r} \int_0^t F(\mathbf{r},\tau)d\tau\right]^N, \tag{31}$$

where $F(\mathbf{r},\tau)$ is the probability density that a walker starting at site $\mathbf{r}$ reaches the origin for the first time at time $\tau$. The integral allows for the first passage of a walker to the origin in the interval $(0,t)$. The factor of $1/V$ enters as the probability of a walker starting at a site, and a sum over all possible starting points for a walker is performed. The bracket calculates the probability that a particular walker has not reached the origin and it is raised to the $N$th power for the probability that none of the walkers has yet reached the origin. The problem is easier in the limit $V$ and $N\to\infty$, with the ratio $c=N/V$ remaining constant. In this limit, we obtain[6]

$$\Phi(t) = \exp\left[-c\sum_l \int_0^t F(l,\tau)d\tau\right] = e^{-cS(t)}, \tag{32}$$

where $S(t)$ is the number of distinct sites a walker visits in time $t$. The simplification involving $S(t)$ was accomplished by noting that any of the sites from which a walk can reach the origin in a time $t$ are exactly the same sites a walker starting at the origin can reach in time $t$. The function $S(t)$ has the following form for both random walk jumps governed by finite $\langle t\rangle$ ($\beta=1$) and infinite $\langle t\rangle$ ($\beta<1$):

$$S(t) \approx e^{-at^\beta}, \tag{33}$$

where $a$ is a constant. We have found that the $\beta<1$ case, which gives a ''stretched exponential,'' is a probability limit distribution in that the result only depends on having an infinite mean waiting time for a defect to jump, and not on the specific form of the waiting time distribution.

## VI. LEVY FLIGHTS AND DRIVES

Let $S_N$ be the sum of $N$ identically distributed random variables with zero mean and variance $\sigma^2$, that is, $S_N=X_1+\cdots+X_N$. This sum constitutes a random walk of $N$ steps. For walks with finite jump variances, the central limit theorem implies that

$$p_N(x) = \lim_{N\to\infty}\text{Prob}\left[x < \frac{S_N}{\sqrt{N}} < x+dx\right]$$
$$= (2\pi N\sigma^2)^{-1/2}e^{-x^2/2N\sigma^2}. \tag{34}$$

The mathematics of probability distributions with infinite moments was investigated by Paul Levy in the 1930's.[3,10,9] If the variance of each jump is infinite, then the variance of $N$ jumps is also infinite. This latter behavior implies that the probability distribution of the sum of $N$ steps should have a similar form as the probability distribution of a single step. This property is similar to the properties of fractals and raises the question of when does the whole [the probability distribution $p_N(x)$ of the sum] look like its parts [the probability distribution of a single step $p(x)$]. Levy found for infinite variance random walks that the Fourier transform of $p(x)$ has the form $\tilde{p}(k)=\exp(-|k|^\beta)$ with $\beta<2$. The Gaussian distribution corresponds to $\beta=2$, and the Cauchy distribution, $f(x)=(2/\pi)(1+x^2)^{-1}$ corresponds to $\beta=1$. This behavior is reminiscent of our analysis of fractal time where noninteger exponents entered and signaled self-similar behavior and infinite moments of a probability distribution. In the present case the moments are spatial rather than temporal.

Let us look at a particular case called the Weierstrass random walk,[11] to illustrate the above discussion. Begin with a random walk on a lattice with the following probabilities for jumps of length $r$, where powers of $b$ give the allowable jump lengths

$$p(r) = \frac{q-1}{2q}\sum_{j=0}^\infty q^{-j}(\delta_{r,b^j}+\delta_{r,-b^j}). \tag{35}$$

According to Eq. (35), jumps an order of magnitude farther (in base $b$) occur an order of magnitude less often (in base $q$). About $q$ jumps of size unity are made (the $j=0$ term) and form a cluster of $q$ sites, before a jump of length $b$ occurs (the $j=1$ term), and about $q$ such clusters of visited sites are formed before a jump of length $b^2$ occurs, and so on until an infinite fractal hierarchy of clusters is formed (see Fig. 1 for a two-dimensional Weierstrass random walk). Compare this to a simple random walk on a square lattice with jumps to nearest neighbor sites (see Fig. 2). To see the mathematical condition necessary for this behavior to occur, we examine the Fourier transform of $p(r)$:
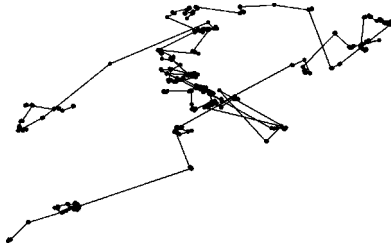
Fig. 1. Typical trajectory of a Levy flight random walk. The mean square jump size is infinite. The flight is between the set of jump points and the lines between points are only a guide to the eye.

$$\tilde{p}(k) = \frac{q-1}{q} \sum_{j=0}^{\infty} q^{-j} \cos(kb^j). \qquad (36)$$

Equation (36) is the famous Weierstrass function which is everywhere continuous, but nowhere differentiable when $b > q$. Note that

$$\langle r^2 \rangle = \sum_{r=-\infty}^{r=+\infty} r^2 p(r) = \frac{q-1}{q} \sum_{j=0}^{\infty} \left( \frac{b^2}{q} \right)^j = -\frac{\partial^2}{\partial k^2} \tilde{p}(k)\big|_{k=0} = \infty, \qquad (37)$$

when $b^2 > q$. This divergence of the second moment (which implies the absence of a length scale) is a sign of fractal self-similar properties. We can write a power series expansion of $\tilde{p}(k)$, but the coefficient of the $\langle r^2 \rangle$ term is infinite. As in the case of a fractal time distribution, a scaling equation is employed:

$$\tilde{p}(k) = \frac{1}{q} \tilde{p}(bk) + \frac{q-1}{q} \cos k. \qquad (38)$$

We can obtain Eq. (38) by using Eq. (36) once for $\tilde{p}(bk)$ and once for $\tilde{p}(k)$ and seeing that their difference is just a single cosine term. The homogeneous part of Eq. (38) has a solution of the form $k^\beta$ with $\beta = \ln q / \ln b$. An infinite series solution for $\tilde{p}(k)$ can be found using Mellin transform techniques.[11] The series expansion has fractional powers, unlike a Taylor series expansion that only has integer powers. For small $k$ (large distances $r$) $\tilde{p}(k)$ behaves as

$$\tilde{p}(k) = e^{-ak^\beta}. \qquad (39)$$

Levy treated a more general case allowing for a bias and additional parameters, but the above discussion captures the spirit. Although $\tilde{p}(k)$ has a simple form, $\exp(-k^\beta)$, explicit forms for $p(x)$ directly in terms of analytic functions are not derivable in general.
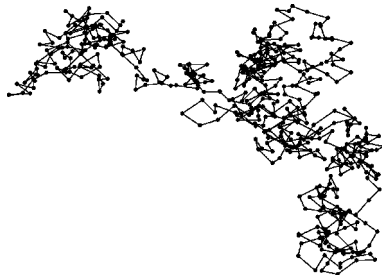


Fig. 2. A typical trajectory of a walker randomly taking nearest neighbor steps on a square lattice.

Despite the beauty of the self-similar trajectories of Levy flights and the scaling equation for $\tilde{p}(k)$, Levy flights were largely ignored in the physics literature until recently[12] because of their infinite moments.[13] This infinity can be tamed by associating a velocity with each flight trajectory segment.[14,15,5,4] We called this random walk with a velocity a *Levy drive* to distinguish it from the *Levy flight* where the walker visits only the end points of a jump and the concept of velocity does not enter. For a single jump in a Levy flight the walker is only at the starting point and the end point, and never in between. For a jump in a Levy drive the walker follows a continuous trajectory between its starting and end points and a finite time is needed to complete the drive. The mean square displacement for a jump in a Levy flight is infinite (and thus not a useful quantity). For a Levy drive the random walker moves with a finite velocity and hence its mean square displacement is never infinite, but it is a time-dependent quantity.

In the continuous-time random walk framework for a Levy drive, we consider $\Psi(r,t)$ to be the probability density of making a jump of displacement $r$ in a time $t$. We write[14]

$$\Psi(r,t) = \psi(t|r)p(r), \qquad (40)$$

or

$$\Psi(r,t) = p(r|t)\psi(t), \qquad (41)$$

where $p(r)$ and $\psi(t)$ have the same meanings as before, and $\psi(t|r)$ and $p(r|t)$ are conditional probabilities for a jump taking a time $t$ given that it is of length $r$, and for a jump being of length $r$ given that it took a time $t$. A simple choice for $\psi(t|r)$ is

$$\psi(t|r) = \delta\left( t - \frac{r}{V(r)} \right). \qquad (42)$$

In the study of turbulent diffusion Kolmogorov assumed a scaling law implying that $V(r) \sim r^{1/3}$. Given this form for $V(r)$ and the Levy flight condition of an infinite second moment,

$$p(r) \approx |r|^{1+\beta}, \qquad (43)$$

with small enough values of $\beta$ so that $\langle r^2 \rangle = \infty$, we find for the mean square displacement,[4]

$$\langle r(t)^2 \rangle = \begin{cases} t^3, & \text{for } \beta \leq 1/3, \\ t^{2+3(1-\beta)/2}, & \text{for } 1/3 \leq \beta \leq 5/3, \\ t & \text{for } \beta \geq 1/3. \end{cases} \qquad (44)$$

The $t^3$ case is called Richardson's law of turbulent diffusion. It corresponds to a Levy drive with Kolmogorov scaling for $V(r)$ combined with a value for $\beta$ such that the mean time spent in a segment between two turning points of the trajectory is infinite. The reason that we did not obtain the usual Levy flight result of $\langle r^2(t) \rangle = \infty$ is that by using the coupled space–time memory $\Psi(r,t)$, we did not calculate $\langle r^2 \rangle = -\partial^2 \tilde{p}(k=0)/\partial k^2$, which would be infinite, but used $\int \exp(ikr)\psi(s|r)p(r)dr$ instead of $\tilde{p}(k)$. In effect, we calculate the distance a walker has gone in a time $t$, instead of the length of the flight segment. The average flight segment can be infinitely long, but at a time $t$ the walker has not completed the flight segment. This type of random walk process has been used to model turbulent pipe flow.[16]

## VII. RANDOM WALKS IN NONLINEAR DYNAMICS

The same type of fractal space and time statistics we have discussed for random systems also appears in dynamical systems, both dissipative and Hamiltonian.[17,13,15,18,19,5,20]

A long-tailed distribution for dynamical phase rotation with a constant rotation rate corresponding to a Levy drive was first presented by Geisel and co-workers.[15] Their investigation involved the diffusion of the chaotic phase (the phase of the wavefunction difference between both sides of the junction) in a Josephson junction using a dynamical map. For a constant voltage across the junction, the phase rotates at a fixed rate, but can change direction intermittently; $N$ complete clockwise rotations correspond to a random walk with a jump of $N$ units to the right, and this jump takes a time proportional to $N$. This jump does not occur instantaneously, but like the hand of a clock there is some time needed to complete a circuit. Mathematically, a jump of length $N$ corresponds to a random drive, and not to a random flight. When the mean square value of $N$ is calculated, it is found to possess Levy drive behavior, that is, it has an infinite second moment. An analysis of the data leads to a map of the form[15]

$$x_{t+1} = (1+\epsilon)x_t + ax_t^z - 1, \tag{45}$$

with $\epsilon$ small. The exponent $z > 1$ makes Eq. (45) nonlinear. Varying $z$ changes the behavior of the mean square displacement. The iterations of the map tend to approach and cluster near $x = 0$ and $x = 1$. This behavior can be interpreted as the system remaining in the same rotation state for long times. The number of iterations clustered in a trajectory near $x = 0$ or $x = 1$ depends on the initial condition. Nearby initial conditions can lead to vastly different numbers of iterations needed to leave these regions. A histogram of the times spent in a rotation state yields a long-tailed distribution just as we have used in random walk problems. The result of a statistical analysis averaged over initial conditions is that[15]

$$\langle r^2(t) \rangle \approx \begin{cases} t^2, & \text{if } z = 2, \\ t^{3-1/(z-1)}, & \text{if } 3/2 < z < 2, \\ t \ln t, & \text{if } z = 3/2, \\ t, & \text{for } 1 < z < 3/2, \end{cases} \tag{46}$$

where $N$ represents the one-dimensional distance from the origin if the rotations were laid out on a line. A positive value of $r = N$ means that there were $N$ more full rotations to the right, than to the left. This behavior corresponds to a constant velocity Levy flight with a long-tailed flight time distribution between reversals of motion with $\psi(t) \approx t^{-1-\beta}$ and $\beta = 1/(z-1)$.

In the above example, we can go directly from $z$, the exponent of the map, to the temporal exponent $\beta = 1/(1-z)$ in the mean square displacement.[15] In most nonlinear dynamical systems the connection between the equations and the kinetics is not obvious or simple. In fact, for the standard map which describes a kicked rotor,[19,17,20]

$$x_{n+1} = x_n + 2\pi K \sin(2\pi\theta_n) \tag{47a}$$

$$\theta_{n+1} = \theta_n + x_{n+1}, \tag{47b}$$

there is no apparent exponent for the dynamics of Eq. (47), but noninteger exponents abound in the description of the orbit kinetics, hinting at a complex dynamics.[13,5] The stan-
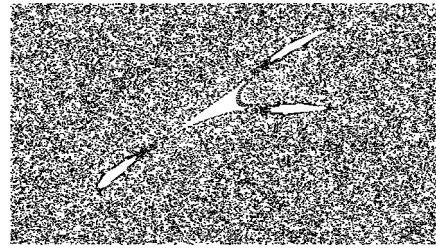


Fig. 3. The trajectory of the standard map with $K = 1.1$ with periodic boundary conditions in both the $x$- and $y$-directions. One sees a period 3 orbit in a chaotic sea. Without the periodic boundary condition in the $y$-direction, the trajectory would perform a random walk in the $y$-direction.

dard map and the Zaslavsky map [see Eq. (48)] have surprisingly complicated dynamics depending on the strength $K$ of the kick.

For $K = 1.1$ for example, a plot of $x$ vs $\theta$ shows some orbits are of the simple closed circular type, while others wander chaotically. Still other orbits exhibit periodicity, that is, $|x_p - x_0| = l$, where a particle moves along the $x$ axis $l$ units in $p$ iterations, with the velocity $l/p$. Sticking close to these orbits provides for the segments of Levy flights, but because we use periodic boundary conditions, it appears that the particle stays in an almost closed orbit and does not enter into chaotic motions. The fractal nature of these island hierarchies, called Cantori because they form a Cantor-like set of tori (see Figs. 3 and 4), creates complex kinetics with Levy probabilities which have been dubbed "strange kinetics."[13] The probability densities for spending time in laminar states near period 3 (for $K = 1.1$) and period 5 (for $K = 1.03$) orbits are long-tailed and have the Levy distribution asymptotic form,[20] $\psi_{\text{period}=3} \approx t^{-2.2}$ and $\psi_{\text{period}=5} \approx t^{-2.8}$. A finer resolution of the period 3 orbit uncovers a period 7 orbit (see Fig. 4), and a finer resolution of one of these orbits would find ten subislands for each period 7 orbit. We would find that islands are composed of sub-islands down to the resolution of the computation.

Predicting these exponents for $\psi(t)$ for these long-tailed distributions is difficult. For example, if $K = 1.1015$ rather than 1.1, the period 3 orbit under further resolution goes to a period 16 orbit, rather than the period 7 orbit. Slightly different $K$ values can give rise to very different exponents because the precise island structure changes in a sensitive manner. Also for different $K$ values the new orbits do not necessarily traverse the hierarchy in the same order. The es-



Fig. 4. Higher resolution plot of the period 3 island of the $K = 1.1$ standard map showing a 7-fold orbit structure. Each of these 7 islands under higher resolution would reveal 10 subislands. This islands within islands structure will continue indefinitely.

cape times from a level of the hierarchy are not a simple function of the distance from the island.[20,17,5]

Another example of Levy drives in dynamical systems is the orbits in the Zaslavsky map, which describes a kicked oscillator,[18,19]

$$u_{n+1} = (u_n + K \sin v_n)\cos \alpha + v_n \sin \alpha, \tag{48a}$$

$$v_{n+1} = -(u_n + K \sin v_n)\sin \alpha + v_n \cos \alpha, \tag{48b}$$

where $\alpha = 2\pi/q$. The orbits form an intricate fractal web, with $q$-fold symmetry throughout the two-dimensional $(u, v)$ phase space. This map only has simple trigonometric nonlinearities, but like the standard map, it requires noninteger exponents for its characterization of trajectories.

As nonlinear and complex systems become better studied, it has been found that very intricate trajectories dominate transport and mixing in these systems. For example, this behavior applies to the standard map and physically this behavior is sought in systems from nonlinear devices to weather patterns. The study of generalizations of random walks in stochastic and nonlinear deterministic systems presents many new problems and opportunities to advance our knowledge of fluctuations beyond the classic Brownian motion work of Einstein.

## APPENDIX: SUGGESTED PROBLEMS

1. Write a program to simulate a simple random walk with constant size jumps of equal probability in the $x$- and $y$-directions. Start the random walker at the center of a circle. Determine how many jumps it takes to exit the circle of radius $a$. Determine how the exit time of the walker depends on the radius of the circle.
2. Write a program to simulate the St. Petersburg game. Make a histogram of your winnings. Determine how your mean winnings increase with the number of times that you play.
3. Compare the shape of the Gaussian and the Cauchy distributions by plotting the distributions on the same graph. Note that the Cauchy distribution has more weight in its tails so larger values are more probable than for the Gaussian.
4. For a probability waiting time density of the form of $\psi(t) = 1/(1 + t^{3/2})$, write a program to calculate a sequence of waiting times. How many jumps occur in a time $T$? Increase $T$ to determine how the number of jumps $N(T)$ depends on $T$. [The answer is $N(T) \propto T^{1/2}$. For a waiting-time density with a finite first moment $N(T)$ should grow as $T$.]
5. Study numerically the standard map, Eq. (47), for $K$ $=1.1$. Find the period 3 orbits in the chaotic sea and then use higher resolution to find the period 7 and period 10 orbits. Determine how far down the hierarchy of islands you can go.
6. Iterate the four-fold symmetric ($q=4$ case) Zaslavsky map, Eq. (48) with $K=6.349972$ and look for Levy-type behavior. See if similar behavior can be found for other $K$ values. Look at other symmetry cases of the Zaslavsky map.

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